Fuzzy Connectivity, Distance Transform, and their Applications

Punam Kumar Saha
Professor
Departments of ECE and Radiology
University of Iowa
pksaha@engineering.uiowa.edu

THE UNIVERSITY OF IOWA

THE IOWA INSTITUTE FOR BIOMEDICAL IMAGING

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[3] PK Saha, JK Udupa, "Fuzzy connected object delineation: axiomatic path strength definition and the case of multiple seeds", Computer Vision and Image Understanding, vol. 83, 275-295, 2001.

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Fuzzy Connectivity and Image Segmentation

- Homogeneity-based affinity
- Object-feature-based affinity
- Object scale: neighborhood size
- Scale-based fuzzy affinity and connectivity
- Evaluation
  - qualitative
  - quantitative
- Conclusion

Motivation

- Connectivity: a popularly used tool for region growing
- Applications: image segmentation, object tracking, object separation
- A fuzzy model for connectivity analysis is essential to capture the global extent of an object using local hanging togetherness and path connectivity

CE-MRA Image data

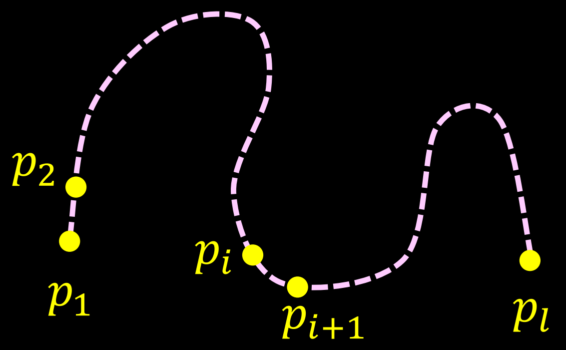
Segmented vasculature

Separated arteries/veins

Separation of arteries and veins in a contrast-enhanced magnetic resonance angiographic (CE-MRA) image data using iterative relative fuzzy connectivity

Image, Adjacency, and Paths

- Image grid: An n-dimensional (nD) cubic grid represented by Z^3 | Z is the set of integers
- Spel: An element of the image grid represented by integer coordinates
- Adjacency: Spatial nearness between two spels
- Image: An n-dimensional grid with image feature value(s) associated with every spel
- Path: A sequence π of spels $\langle p_1, p_2, \cdots p_l \rangle$ where every two successive spels p_i, p_{i+1} are adjacent



A path π

Affinity and Strength of a Path

Fuzzy Affinity (κ): local hanging-togetherness between two spels

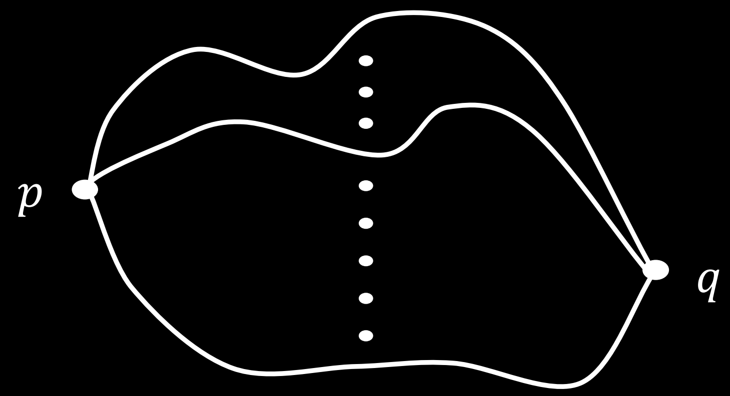
- $\kappa(p,q) \in [0,1]$
- $\kappa(p,q)$ is zero if p, q are non-adjacent
- $\kappa(p,p) = 1$, i.e., reflexive
- $\kappa(p,q) = \kappa(q,p)$, i.e. symmetric

Strength (Π) of a path (π = $\langle p_1, p_2, \cdots p_l \rangle$)

- $\Pi(\pi)$ = the affinity of the weakest link on the path, i.e.,

$$\Pi(\pi) = \min_{1 \le i < l} \kappa(p_i, p_{i+1})$$

Fuzzy Connectivity



Fuzzy Connectivity (K): Strength of the strongest path between two spels

- $\mathrm{K}(p, q)$ = strength of the strongest path between two spel, i.e.,

$$\mathrm{K}(p, q) = \max_{\pi \in \mathcal{P}_{p,q}} \Pi(\pi)$$

$\mathcal{P}_{p,q}$ is the set of all possible paths between p and q.

Fuzzy Connectivity: Properties

Theorem 1. For any image $\mathcal{C} = (C, f)$ over (Z^n, α), and for any affinity κ, fuzzy connectivity K in $\mathcal{C}$ is a similitude relation in C if and only if

$$\mathrm{K}(p, q) = \max_{\pi \in \mathcal{P}_{p,q}} \left[\min_{1 \le i < l} \mu_\kappa(p_i, p_{i+1} \right],$$

where π is the path $\langle p_1, p_2, \cdots p_l \rangle$.

Theorem 2. For any image $\mathcal{C} = (C, f)$ over (Z^n, α), and for any affinity κ, and for any $\theta \in [0,1]$, the $\kappa\theta$-object $O_{K\theta}(S)$ of $\mathcal{C}$ containing S is

$$\mathrm{O}_{\mathrm{K}\theta}(\mathrm{S}) = \left\{ c \mid c \in C \text{ and } \max_{s \in S} [K(s, c)] \ge \theta \right\}.$$

Theorem 3. For any image $\mathcal{C} = (C, f)$ over (Z^n, α), and for any affinity κ, for any $\theta \in [0,1]$, and for any two no-empty sets $S_1, S_2 \in C$, the $\kappa\theta$-objects $O_{K\theta}(S_1)$ and $O_{K\theta}(S_2)$ are equal if and only if $\mathrm{S}_1 \subset O_{K\theta}(S_2)$ and $\mathrm{S}_2 \subset O_{K\theta}(S_1)$.

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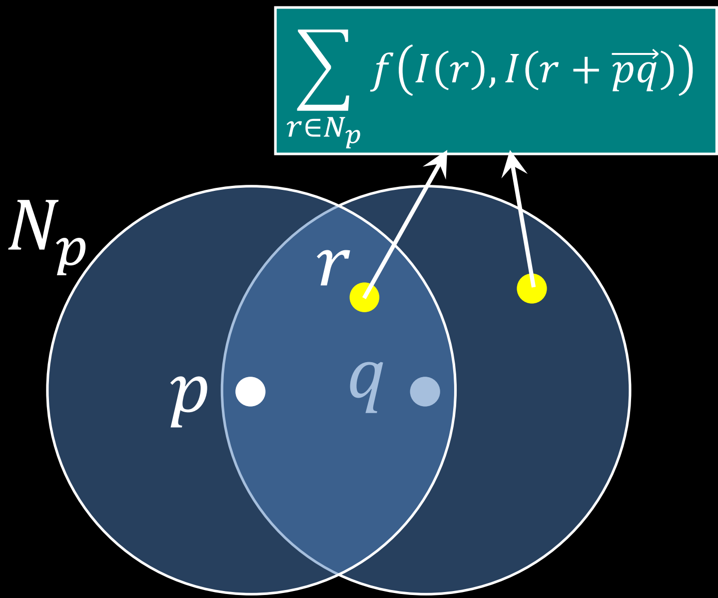
- Saha, Udupa, “Fuzzy connected object delineation: axiomatic path strength definition and the case of multiple seeds”, Comp Vis Imag Und, **83**: 275-295, 2001.
- Udupa, Saha, "Fuzzy connectedness and image segmentation," Proceedings of the IEEE, **91**: 1649-1669, 2003

Scale-Based Fuzzy Affinity and Connectivity

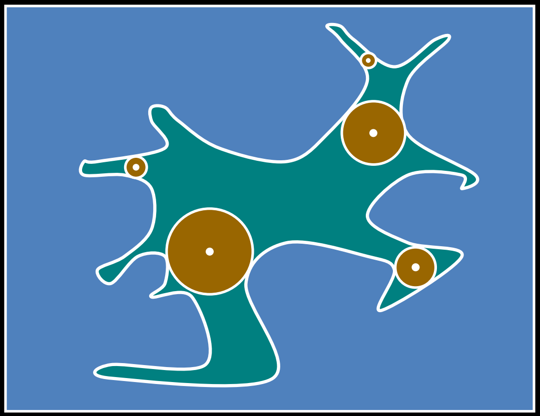
Notion of Local Scale in Fuzzy Affinity



A generic approach of computing the affinity or local hanging-togetherness between two nearby points p, q.

Q. What is the size of neighborhood N_p at a point p?

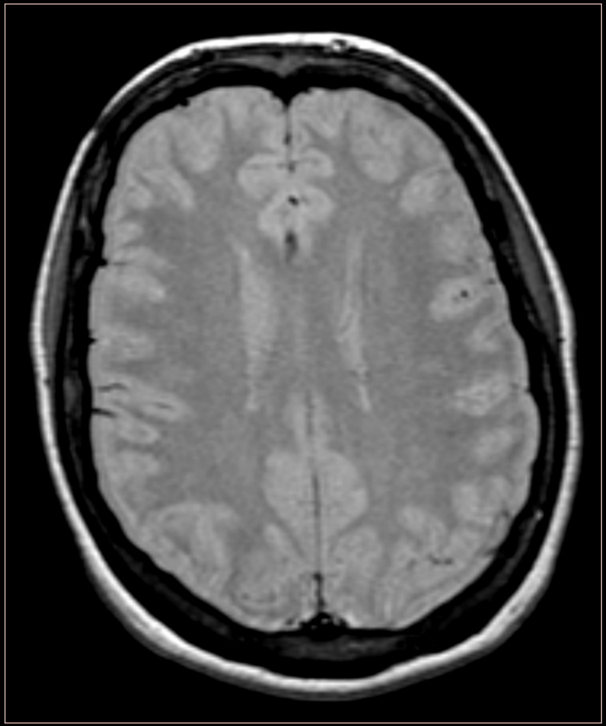
Scale: size of the largest hyper-sphere entirely contained in the object region determines the local scale or the neighborhood size.



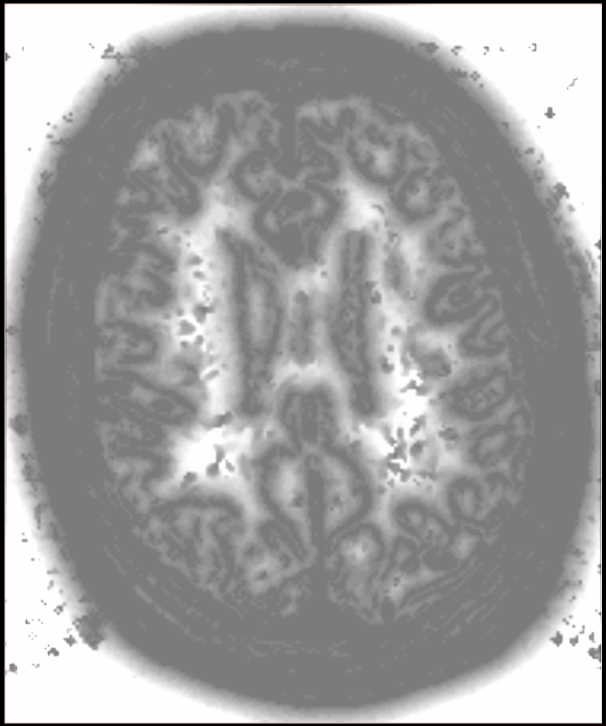
- Saha, Udupa, Odhner, "Scale-based fuzzy connected image segmentation: theory, algorithms, and validation," Comp Vis Imag Und, **77**: 145-174, 2000

Computation of Scale

A method is developed for computing scale at every location without image segmentation.



original

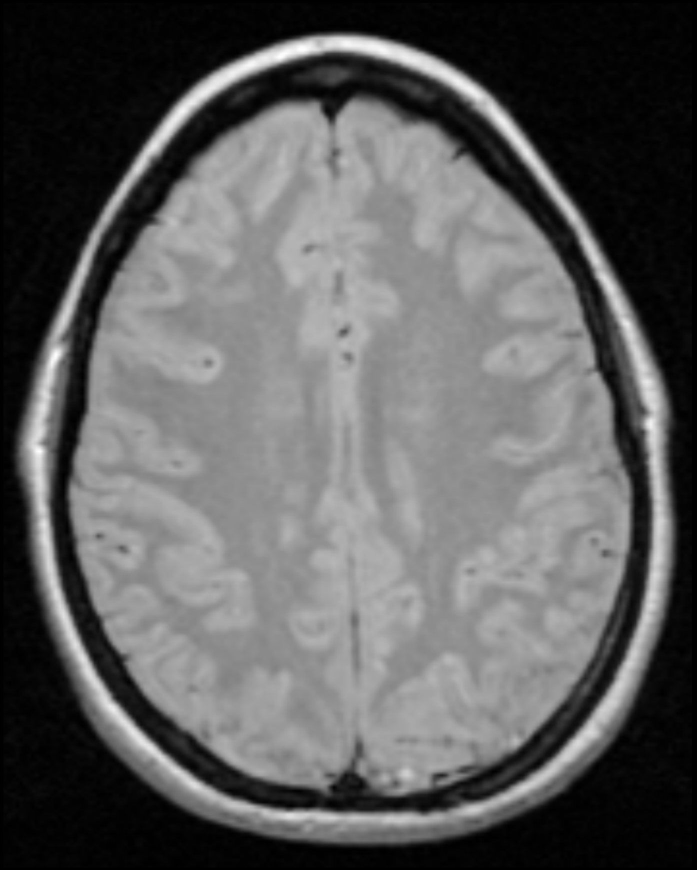


scale

Scale-Based Affinity

Aspects

- Spatial nearness (μ_α)
- Homogeneity-based affinity (μ_h)
- Object-feature-based affinity (μ_o)

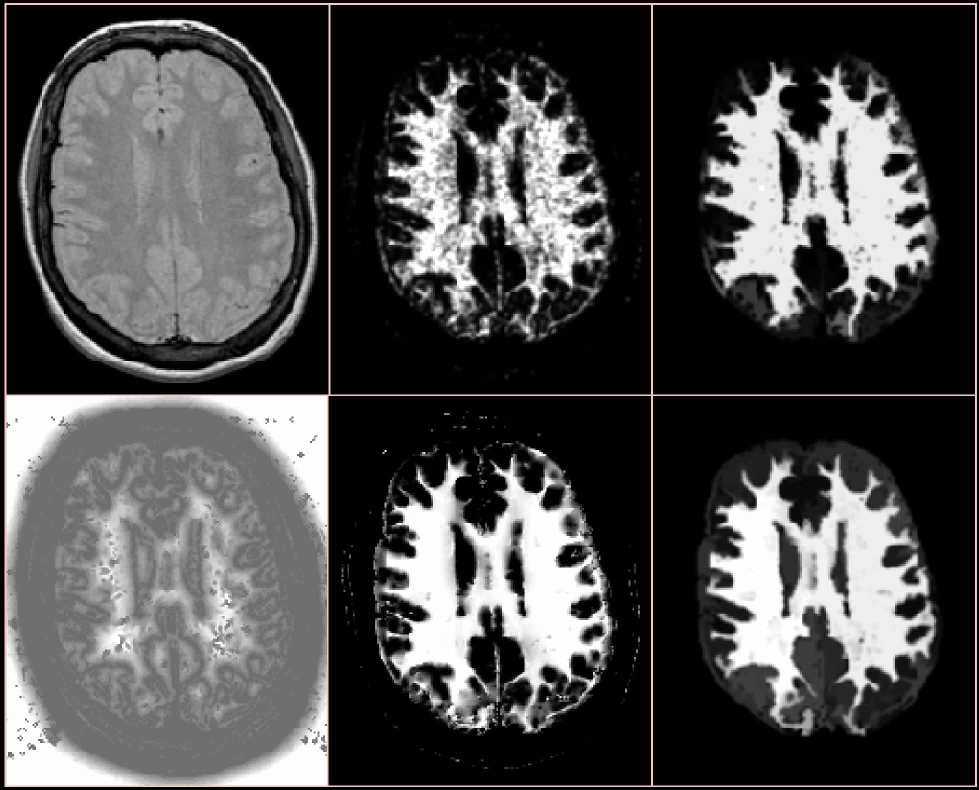


Combined affinity

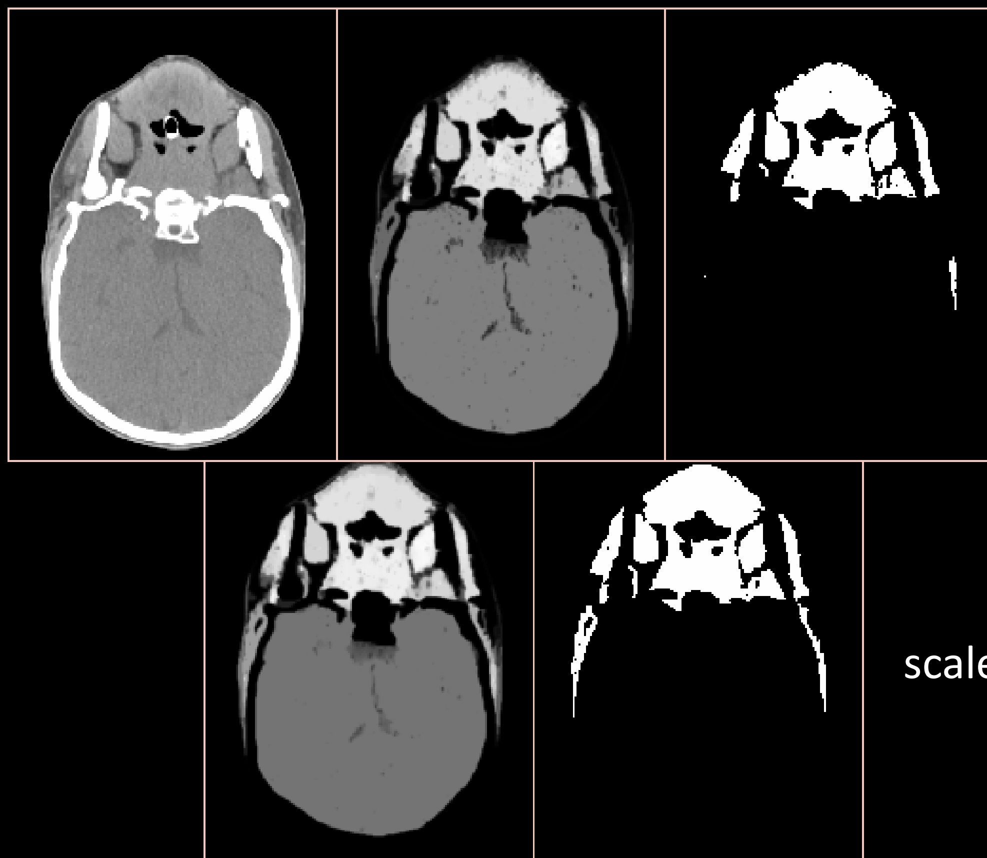
$$\mu_\kappa = \mu_\alpha \sqrt{\mu_h \mu_o}$$

- Saha, Udupa, Odhner, "Scale-based fuzzy connected image segmentation: theory, algorithms, and validation," Comp Vis Imag Und, **77**: 145-174, 2000

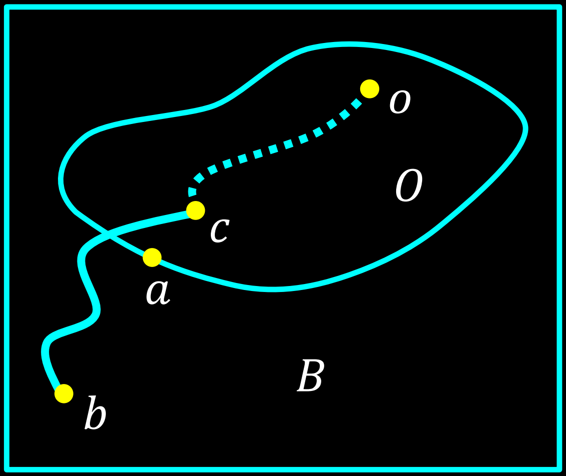
Results of Scale-Based Affinity



Results of Scale-Based Affinity



Relative Fuzzy Connectivity



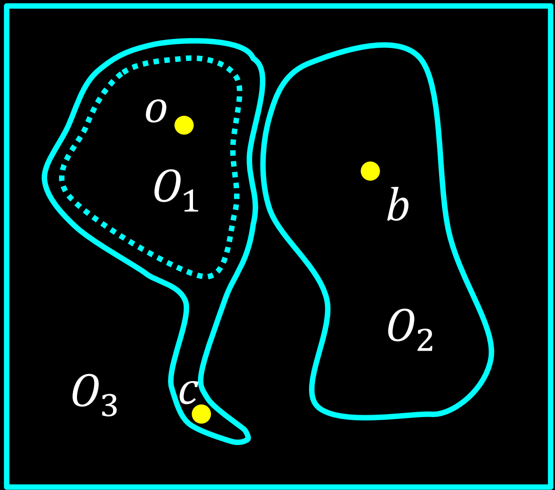
- The key idea of relative fuzzy connectedness is to consider all co-objects of importance that are present in an image and to let them to compete among themselves in having image locations as their members.
- Let o and b represent the seed of two different objects.
- A point c is assigned to the object whose seed is more strongly connected to c.
- The object P_{ob_κ} represented by the seed o relative to the object represented by the seed b is

$$P_{ob_\kappa} = \{c \mid c \in C \text{ and } K(o,c) > K(b,c)\}$$

- Saha, Udupa, "Relative fuzzy connectedness among multiple objects ….. in image segmentation," Comp Vis Imag Und, **82**: 42-56, 2001.
- Udupa, Saha, Lotufo, "Relative fuzzy connectedness and …….. in image segmentation," IEEE Trans Pat Anal Mach Intel, **24**: I485-1500, 2002.

Iterative Relative Fuzzy Connectivity

- The principle behind this strategy is to iteratively refine the competition rules for different objects depending upon the results of the previous iteration.
- Due to blurring where the two objects O_1, O_2 come close, it is likely that a point c has $K(o,c) = K(b,c)$.
- In such a situation, the best path from b to c need to pass through the core of a.
- Iterative relative fuzzy connectivity



$$\forall p,q \in C, \qquad \mu_{\kappa_{ob}}^{0}(p,q) = \mu_\kappa(p,q)$$
$$P_{ob_\kappa}^{0} = \{c \mid c \in C \text{ and } K^0(o,c) > K^0(b,c)\}$$

Initial rules

$$\forall p,q \in C, \qquad \mu_{\kappa_{ob}}^{0}(p,q) = \begin{cases} 0, & \text{if } p \text{ or } q \in P_{ob_\kappa}^{i-1}, \\ \mu_\kappa(p,q), & \text{otherwise.} \end{cases}$$
$$P_{ob_\kappa}^{i} = \{c \mid c \in C \text{ and } K^i(o,c) > K^i(b,c)\}$$

Iterative refinement

- Saha, Udupa, "Iterative relative fuzzy connectedness …… in image segmentation," Proc of the IEEE Workshop on Math Meth Biomed Imag Anal (MMBIA), 28-35, 2000.
- Udupa, Saha, Lotufo, "Relative fuzzy connectedness and …….. in image segmentation," IEEE Trans Pat Anal Mach Intel, **24**: I485-1500, 2002.
- Ciesielski, Udupa, Saha, Zhuge, "Iterative relative fuzzy connectedness for multiple objects with multiple seeds," Comp Vis Imag Und, **107**: 160-182, 2007

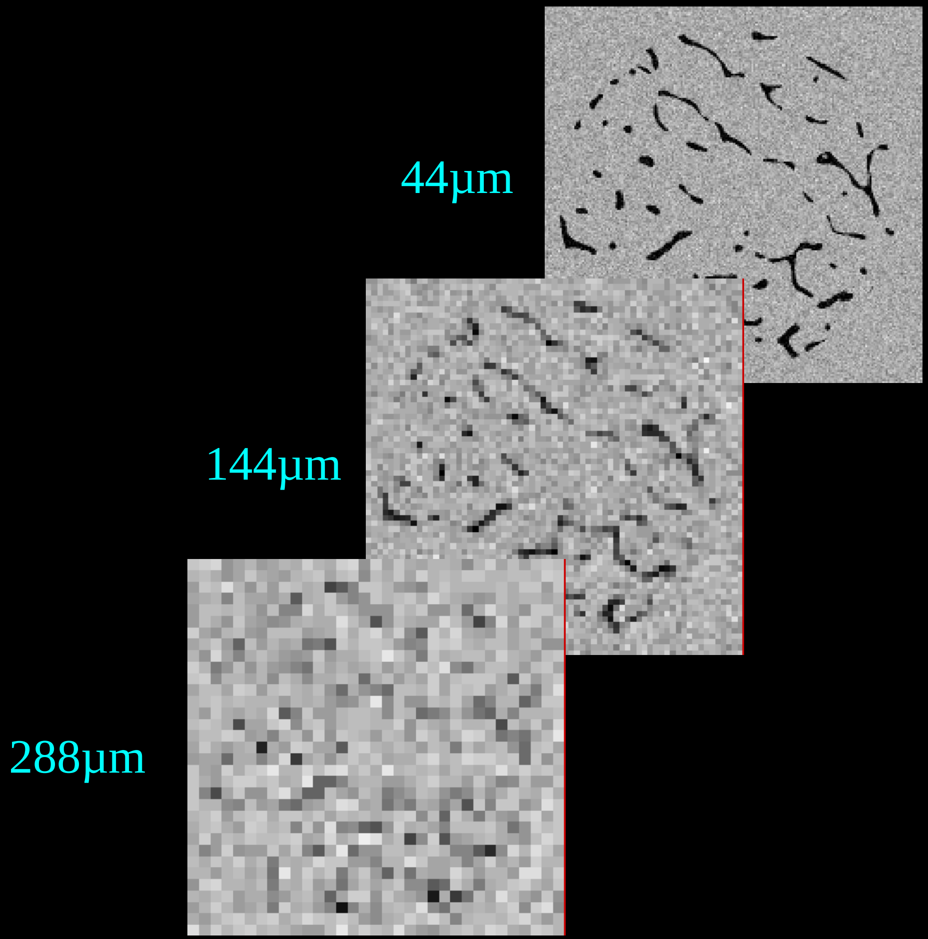
Summary

- Interpretation of object scale as neighborhood size.
- Computation of scale without explicit object segmentation
- A new class of scale-based fuzzy affinity relations considering both homogeneity and object-features
- Relative fuzzy connectedness simultaneously grows multiple objects
- Iterative relative fuzzy connectedness refines the rules of competition to iteratively separate multiple objects starting a large scale and iteratively progressing to finer details

Fuzzy Distance Transform

Motivation

- Distance transform is a popularly used tool for object shape analysis
- Applications: feature extraction, local thickness or scale computation, skeletonization, morphological and shape-based object analysis
- A fuzzy model for distance-based analysis is essential in a limited resolution regime.



MR image of human trabecular bone at different resolution regimes

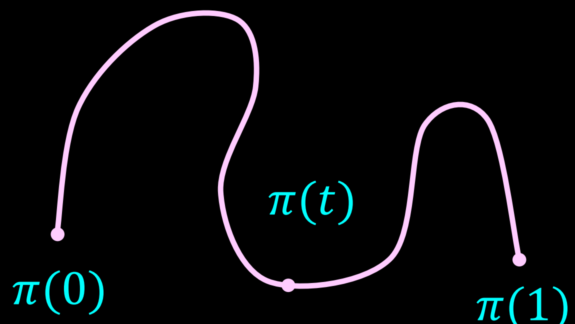
Fuzzy Subsets, Objects, and Paths

- A fuzzy subset: a set of pairs

  location, membership value (μ)

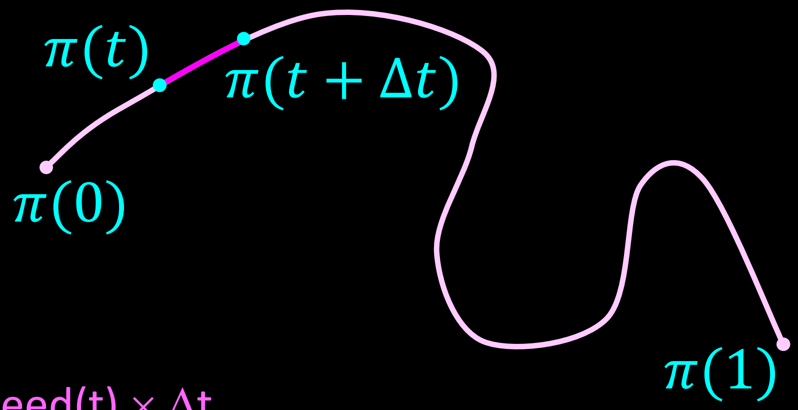
- Support of an object: locations with non zero membership

- A path π is a continuous function (an walk) of time $t \in [0,1]$



Length of a Path

Distance. The minimum material that has to be traversed through to proceed from one point to the other



Δfuzzy length(t) = material(π(t)) × Δarc length(t)

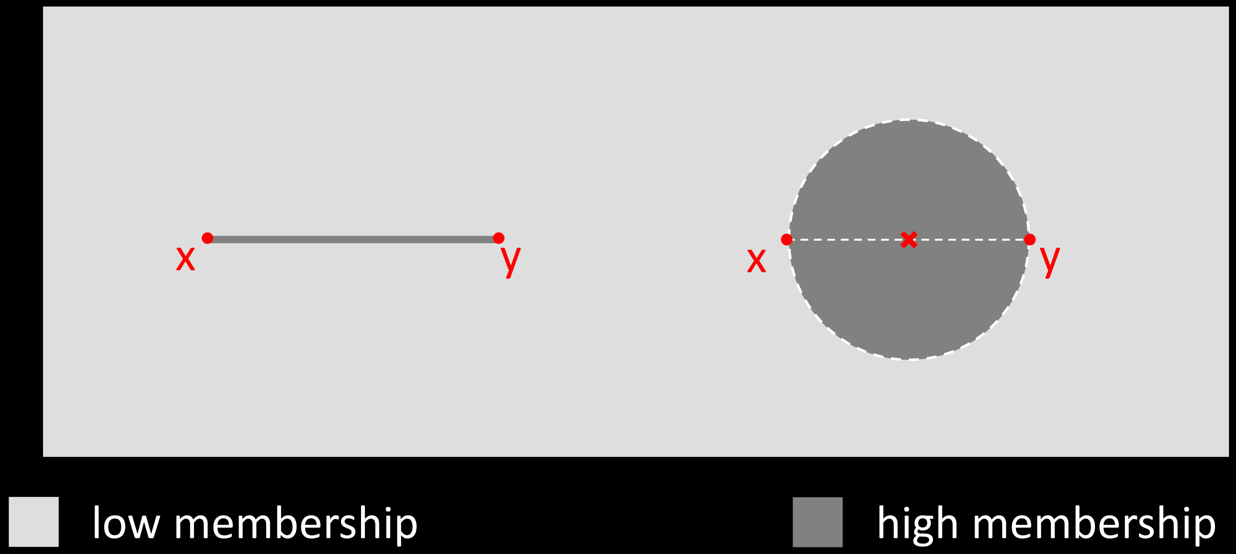
Δarc length(t) = speed(t) × Δt

Fuzzy length of a path

$$\Pi(\pi) = \int_0^1 \mu(\pi(t)) \left| \frac{d\pi(t)}{dt} \right| dt$$

- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002

Properties of the Shortest Path

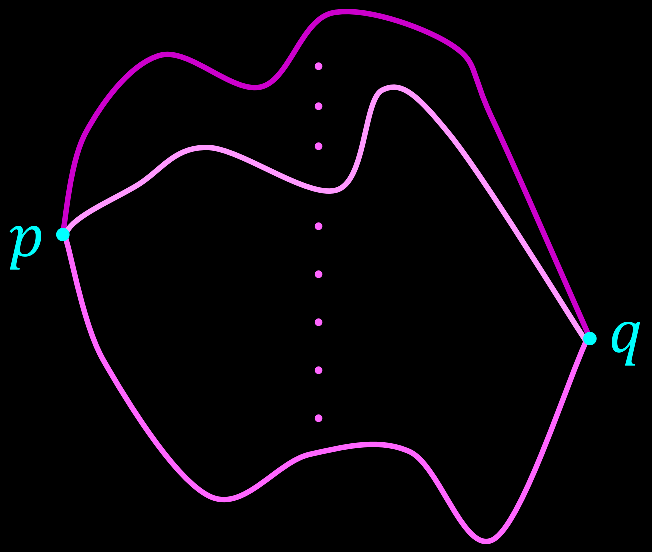


- Existence of the shortest path is not guaranteed
- There may be multiple shortest path
- The shortest path may not be a straight line segment even for a convex fuzzy subset set

Fuzzy Distance Transform

- Fuzzy distance between two points is the infimum of fuzzy lengths of all paths between them.

- Fuzzy distance transform (FDT) at a point is the infimum of the distances the candidate point and a point in the inverse of the support.



- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002

Metric Properties of Fuzzy Distance

Theorem 1. For any fuzzy subset of $\Re^n$, fuzzy distance is a metric for interior of the support of the fuzzy subset.

- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002

Fuzzy Distance in Generalized Digital Space

- Digital grid G in $\Re^n$ is a locally finite set of points over $\Re^n$
- The distance between any two points $p, q \in G$ is bounded by $\Delta_{\min}$ and $\Delta_{\max}$

- A Digital space D is an ordered pair (G, α), where
- α is a binary adjacency relation on G
- For all $p \in G$, $\alpha(p, q)$ is nonzero for finitely many qs, only.

- A Digital object $\mathcal{O}$ a fuzzy subset of G
- $\mu_{\mathcal{O}}: G \rightarrow [0,1]$ is the membership function of $\mathcal{O}$
- $\Theta(\mathcal{O}) = \{p \mid p \in G \text{ and } \mu_{\mathcal{O}} \neq 0\}$ is the support of $\mathcal{O}$

- Saha, Wehrli, “Fuzzy distance transform in general digital grids and its applications”, Proc of 7th Joint Conf Info Sc, Research Triangular Park, NC, 201-213, 2003

Digital Paths and Links

A digital path $\pi = \langle p_1, p_2, \cdots, p_l \rangle$ is a sequence of adjacent points, i.e., $\forall 1 \le i < l,\ \mu_\alpha(p_i, p_{i+1}) = 1$

A link $\langle p, q \rangle$ is an digital path of exactly two points

Length $\mu_{d_\mathcal{O}}(\langle p, q \rangle)$ of a link $\langle p, q \rangle$

- length of any link $\langle p, q \rangle$ is nonnegative
- the length of a link $\langle p, p \rangle$is always 0;
- the lengths of the links $\langle p, q \rangle$ and $\langle q, p \rangle$ are equal;
- the length of the link is nonzero if $p \in \Theta(\mathcal{O})$ and $p \ne q$

$$\mu_{d_\mathcal{O}}(\langle p, q \rangle) = \frac{1}{2}(\mu_\mathcal{O}(p) + \mu_\mathcal{O}(p)) \times \|p - q\|$$

- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002
- Saha, Wehrli, “Fuzzy distance transform in general digital grids and its applications”, Proc of 7th Joint Conf Info Sc, Research Triangular Park, NC, 201-213, 2003

Path Length

Length $\Pi_{\mathcal{O}}(\pi)$ of a path $\pi = \langle p_1, p_2, \cdots, p_l \rangle$ in is the sum of the lengths of all links along the path, i.e.

$$\Pi_{\mathcal{O}}(\pi) = \sum_{i=1}^{l-1} \mu_{d_{\mathcal{O}}}(\langle p_i, p_{i+1} \rangle)$$

Theorem 1. For any digital space $D = (G, \alpha)$, for any fuzzy object $\mathcal{O}$ on D with a bounded support, for any two points $p, q \in G$, there exists a shortest path between them.

- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002
- Saha, Wehrli, “Fuzzy distance transform in general digital grids and its applications”, Proc of 7th Joint Conf Info Sc, Research Triangular Park, NC, 201-213, 2003

Fuzzy Distance

Fuzzy distance $\omega_{\mathcal{O}}(p,q)$ between two grid points $p,q,\in G$ is the length of the shortest path from p to q, i.e.,

$$\omega_{\mathcal{O}}(p,q) = \min_{\pi \in \mathcal{P}_{p,q}} \Pi_{\mathcal{O}}(\pi)\,,$$

where, $\mathcal{P}_{p,q}$ is the set of all paths from p to q

Theorem 2. For any digital space $D = (G,\alpha)$, for any fuzzy object $\mathcal{O}$ on D with a bounded support, for valid link-length function $\mu_{d_{\mathcal{O}}}$, fuzzy distance $\omega_{\mathcal{O}}$ is a metric for the support $\Theta(\mathcal{O})$ of $\mathcal{O}$.

- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002
- Saha, Wehrli, “Fuzzy distance transform in general digital grids and its applications”, Proc of 7th Joint Conf Info Sc, Research Triangular Park, NC, 201-213, 2003

Fuzzy Distance Transform

Fuzzy distance transform (FDT) $\Omega_{\mathcal{O}}(p)$ at any point $p \in G$ the fuzzy distance between p and a nearest points in the complement of the support $\Theta(\mathcal{O})$ of $\mathcal{O}$, i.e.,

$$\Omega_{\mathcal{O}}(p) = \min_{q \in \Theta(\mathcal{O})} \omega_{\mathcal{O}}(p, q)$$

- Saha, Wehrli, Gomberg, "Fuzzy distance transform: theory, algorithms, and applications," Comp Vis Imag Und, **86**: 171-190, 2002
- Saha, Wehrli, “Fuzzy distance transform in general digital grids and its applications”, Proc of 7th Joint Conf Info Sc, Research Triangular Park, NC, 201-213, 2003

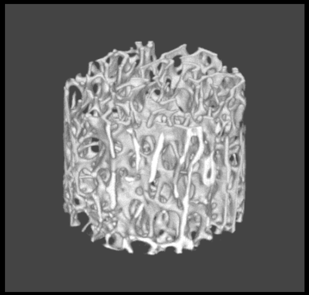
Algorithmic Solution for FDT

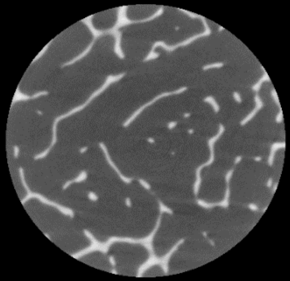
- FDT is computed using a dynamic programming based method (DP).
- DP for computing FDT terminates in a finite number of steps.
- On termination, DP accurately computes FDT.

FDT-Based Thickness Computation

Performance under varying image resolution: trabecular bone model

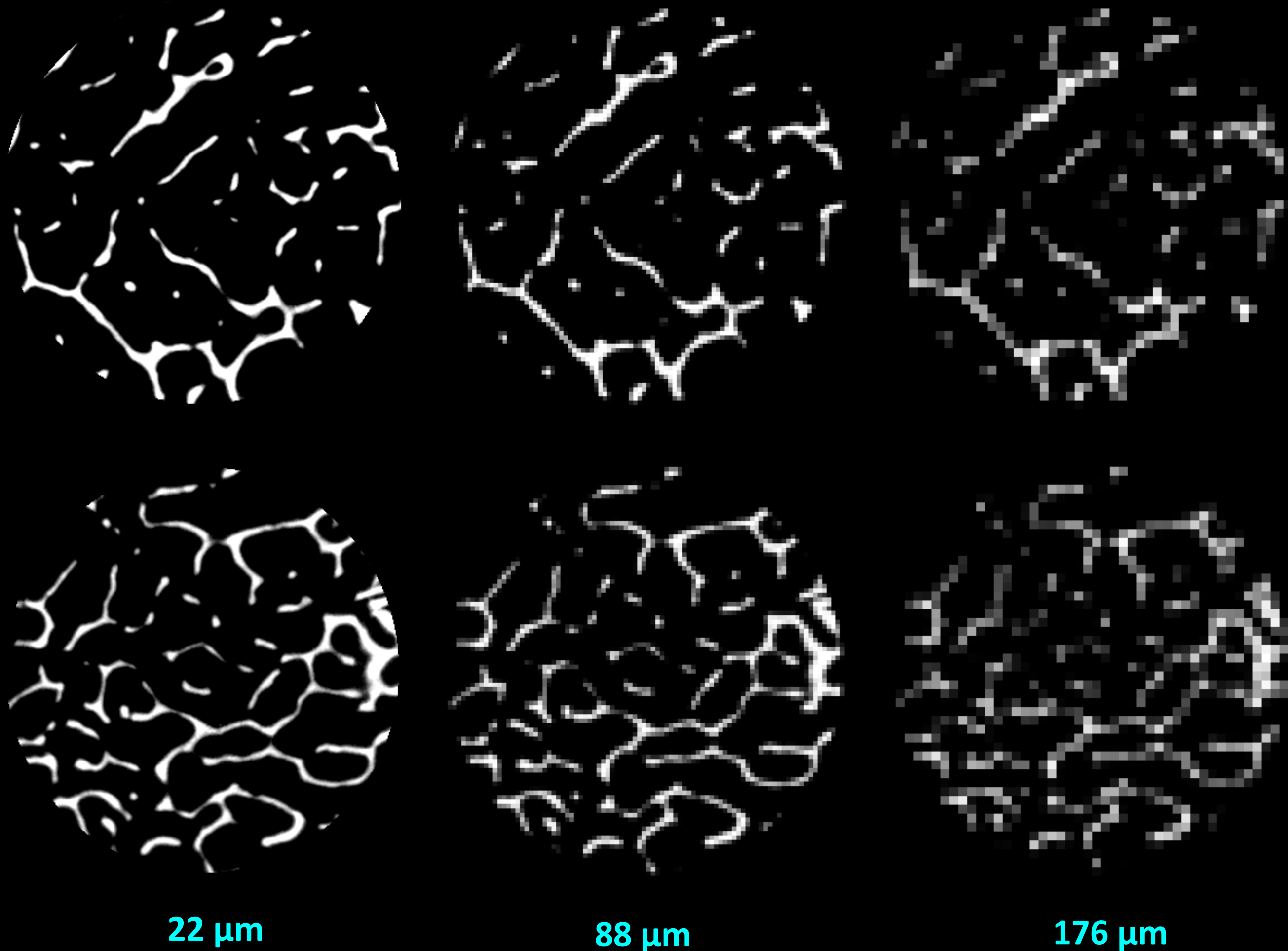
- Image acquisition
  - Two cadaveric distal radius samples
  - Cylindrical cores of 9 mm height and 9 mm diameter with the cylinder axis parallel to the direction of the radius
  - SANCO Medical μ-CT 20 scanner
  - 22 μm isotropic voxel
- Preprocessing
  - BVF Mapping
- FDT computation
- Thickness computation
  - 3D skeletonization
  - Sampling FDT values along skeleton



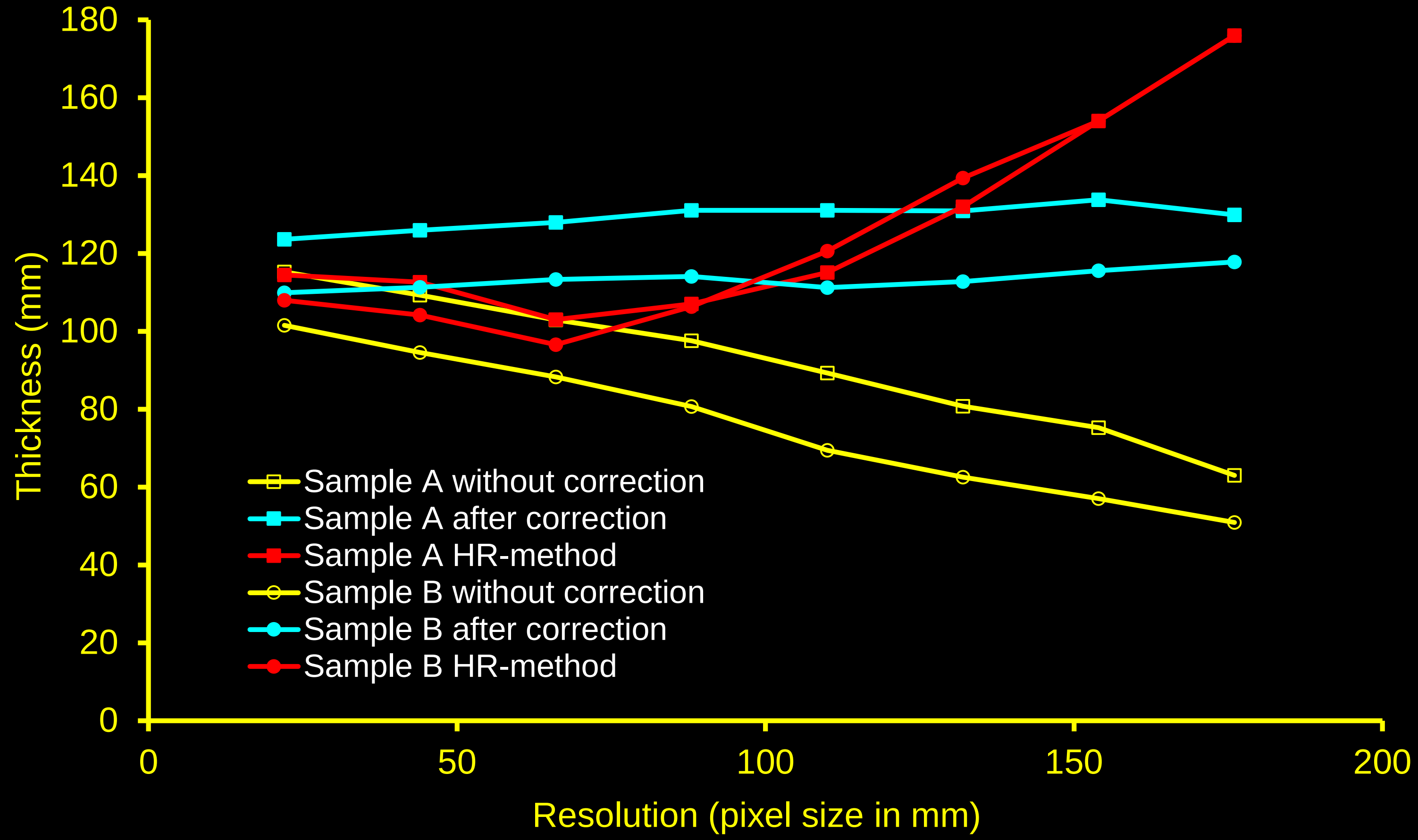


- Saha, Wehrli, "Measurement of trabecular bone thickness …… in vivo MRI by fuzzy distance transform," IEEE Trans Med Imag, **23**: 53-62, 2004

Images at Different Resolutions



Results



- Saha, Wehrli, "Measurement of trabecular bone thickness …… in vivo MRI by fuzzy distance transform," IEEE Trans Med Imag, **23**: 53-62, 2004

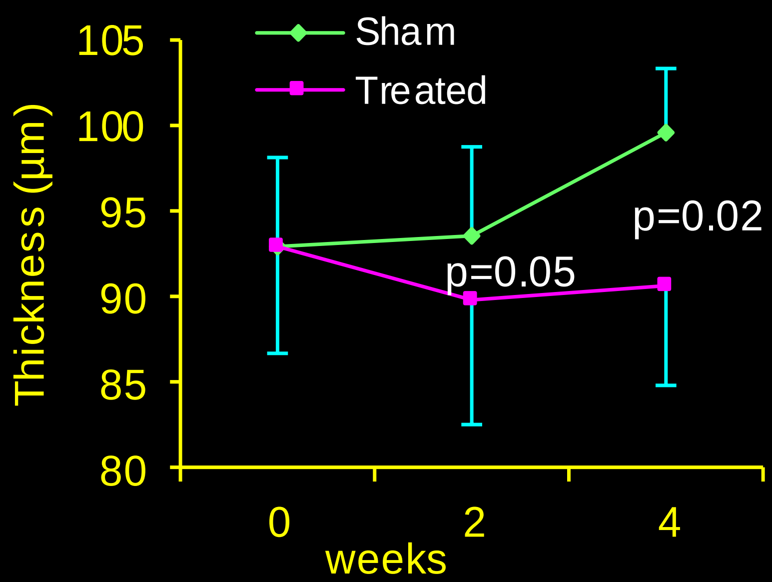
Longitudinal Study Showing Effects of Treatment: A Rabbit Model

- NZW Rabbit (3.5-4 Mo. Old; N=12)

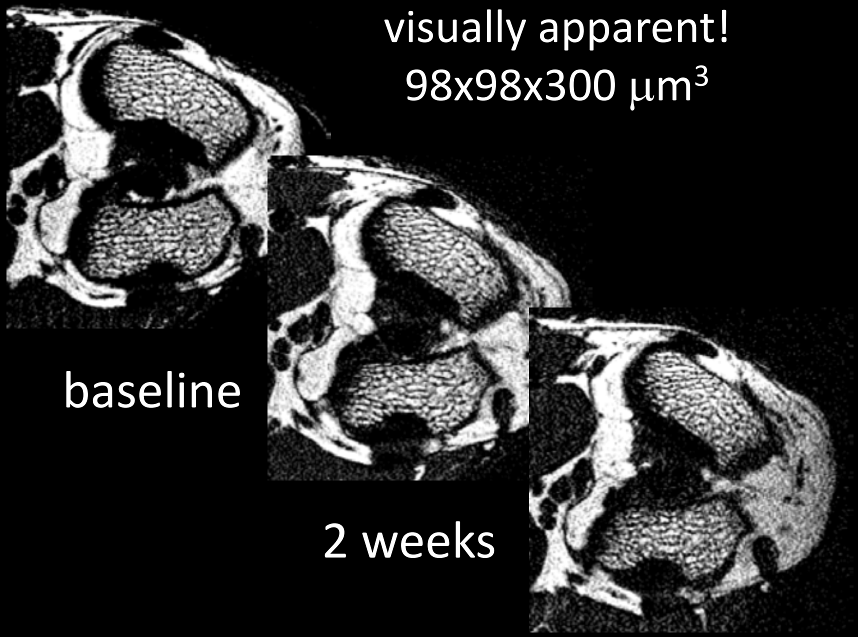
- Sham Operation (n=6)
- 0.4 mg/kg/day Dexamethazone (n=6)

- MR image acquisition (Voxel size 98x98x300 μm^3)
  - Baseline
  - 2 weeks
  - 4 weeks

Results



Treatment effects are not visually apparent!
98x98x300 μm^3



baseline

2 weeks

4 weeks

Effect of dexamethasone on rabbit trabecular thickness.

- Saha, Wehrli, "Measurement of trabecular bone thickness …… in vivo MRI by fuzzy distance transform," IEEE Trans Med Imag, **23**: 53-62, 2004

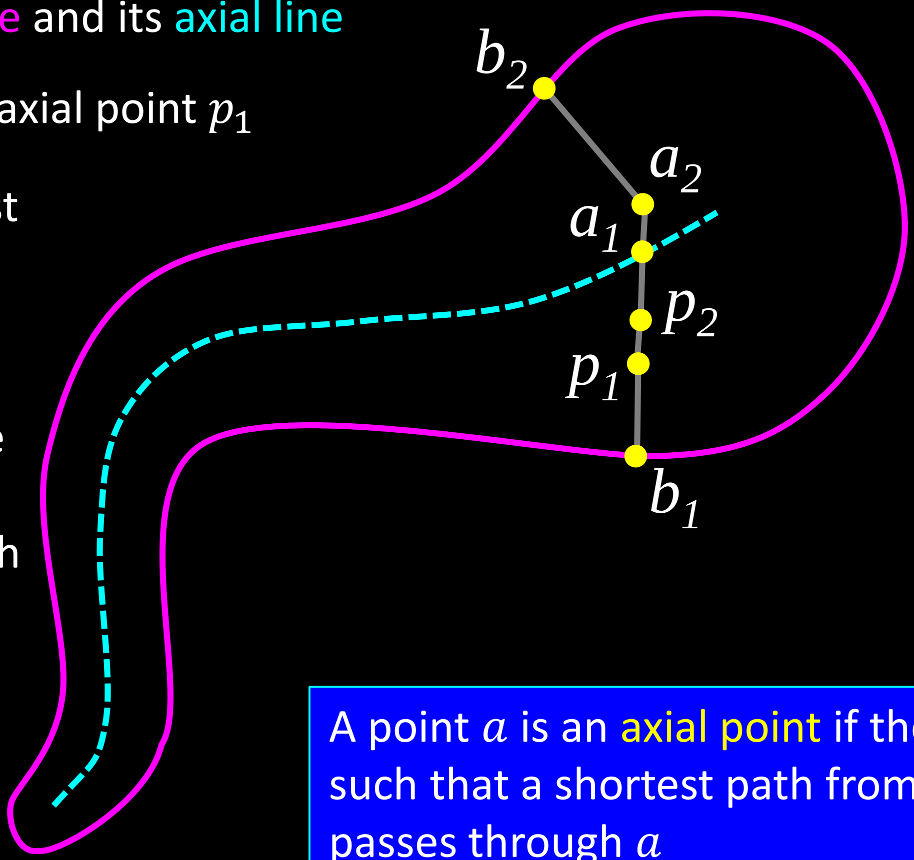
FDT-Based Detection of Axial Points

Fuzzy skeletonization yields a compact representation directly from the fuzzy representation of an object

A major challenge in fuzzy skeletonization is to recognize the axial points (often referred to as shape points or representative points) that in some sense describes the shape of the skeleton (but not the topology)

How to Locate an Axial Point

1. Consider a shape and its axial line
2. Consider a non-axial point p_1
3. Find the shortest path $p_1 b_1$ to boundary
4. If we extend the path $p_1 b_1$ to p_2 the shortest path $p_2 b_1$ passes through p_1



5. Now, consider an axial point a_1
6. If extend the shortest path $a_1 b_1$ to a_2, the shortest path from a_2 to the boundary does not passes through a_1

A point a is an axial point if there is no point a' such that a shortest path from a' to the boundary passes through a

Fuzzy Axial Points

- A point p is an axial point if there is no point p' such that a shortest path from p' to the boundary passes through p.
- Specifically, a point p is an axial point if there is no point q in the neighborhood of p such that

$$\Omega_{\mathcal{O}}(q) = \Omega_{\mathcal{O}}(p) + \mu_{d_{\mathcal{O}}}(p, q)$$

- Arcelli and Sanniti di Baja introduced a criterion to detect the centers of maximal balls (CMBs) in a binary digital image using 3×3 weighted distance transform
- Borgefors extended it to 5×5 weighted distances
- This concept was generalized to fuzzy sets by Saha and Wehrli, Svensson, and Jin and Saha

- Arcelli, Sanniti di Baja, “Finding local maxima in a pseudo-Euclidean distance transform”, Comput Vis Graph Imag Proc, 43: 361-367, 1988
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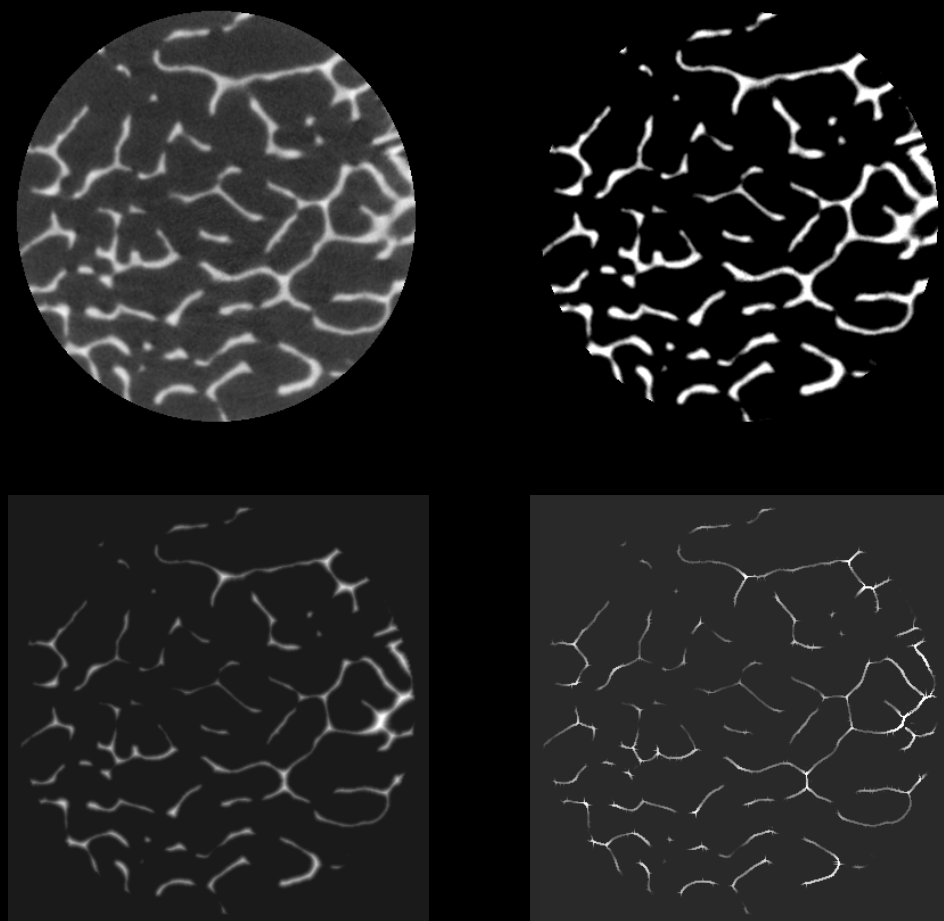
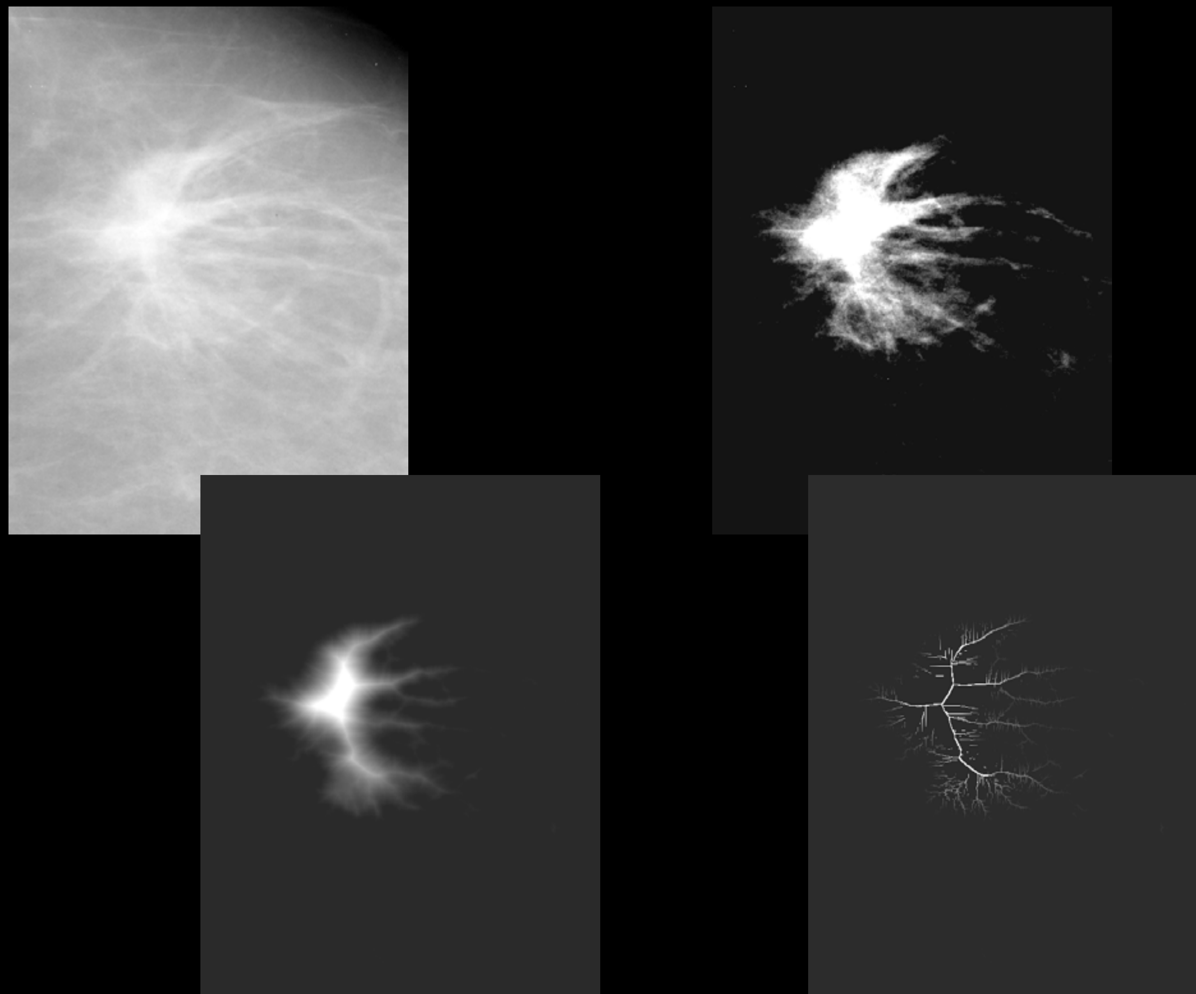
- Saha, Wehrli, “Fuzzy distance transform in general digital grids and its applications”, Proc of 7th Joint Conf Info Sc, Research Triangular Park, NC, 201-213, 2003
- Svensson, “Aspects on the reverse fuzzy distance transform”, Patt Recog Lett **29**: 888-896, 2008
- Jin, Saha. "A new fuzzy skeletonization algorithm and its applications to medical imaging." Proc. of 17th Int Conf on Imag Anal Proc (ICIAP), 662-671, 2013.

Results



Results



Results

Summary

- The theory of fuzzy distance has been extended to a more general grid
- The conditions for valid length functions for links have been established
- Accuracy of FDT-based thickness computation has been studied and its superiority over the hard DT-based method at various resolutions has been experimentally observed.
- A new FDT-based method is presented to directly recognize axial points in the fuzzy representation of an object and the results of different applications have been presented